Calculate the limit

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$$\lim_{n\to\infty} \ln\left(\frac{1}{2^n}\prod_{k=1}^{n}\left(2+\frac{k}{n^2}\right)\right)$$

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Let $a_n := \ln\left(\frac{1}{2^n}\prod_{k=1}^{n}\left(2+\frac{k}{n^2}\right)\right)$. Since $a_n = \ln\left(\prod_{k=1}^{n}\left(1+\frac{k}{2n^2}\right)\right) = \sum_{k=1}^{n}\ln\left(1+\frac{k}{2n^2}\right)$,

$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}, \sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$ and $x - \frac{x^2}{2} < \ln(1+x) < x,\ \forall x > 0$

then $\sum_{k=1}^{n}\left(\frac{k}{2n^2} - \frac{k^2}{8n^4}\right) < a_n < \sum_{k=1}^{n}\frac{k}{2n^2} \Leftrightarrow \frac{n+1}{4n} - \frac{(n+1)(2n+1)}{48n^3} < a_n < \frac{n+1}{4n}$.

and, therefore, by Squeeze Principle $\lim_{n\to\infty} a_n = \frac{1}{4}$.